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ABSTRACT

Accurate and automatic production of the basic number combinations is a major objective of elementary mathematics education. Typically, it is not an objective that is easily and quickly attained. Indeed, teachers regularly lament about how difficult it is to get their students to master the basic "number facts." This problem may be due, in part, to educators' misconceptions of how children learn basic number combinations and how number combinations are represented in adult long-term memory. This paper first outlines the historical debate on how number combinations are learned or internalized and then critically reviews the empirical evidence for and conceptual adequacy of current models of how number combinations are represented in and efficiently produced from long-term memory. An alternative view is offered that argues that, while adults may retrieve some number combinations from associative memory (a "reproductive" process), many combinations can be accurately and automatically produced from stored rules, algorithms, or principles (efficient "reconstructive" processes). (JN)

Mastery of the Basic Number Combination: Internalization of Relationships or Facts?

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Mastery of the Basic Number Combinations:

Internalization of Relationships or Facts?

Accurate and automatic production of the basic number combinations is a major objective of elementary mathematics education.[1] Typically, it is not an objective that is easily and quickly attained. Indeed, teachers regularly lament about how difficult it is to get their pupils to master the basic "number facts." This "problem" may be due, in part, to educators' misconceptions of (a) how children learn the basic number combinations and (b) how number combinations are represented in adult long-term memory. This paper first outlines the historical debate on how number combinations are learned or internalized. Then the paper critically reviews the empirical evidence for and conceptual adequacy of current models of how number combinations are represented in and efficiently produced from long-term memory. An alternative view is offered that argues that, while adults may retrieve some number combinations from associative memory (a reproductive process), many combinations can be accurately and automatically produced from stored rules, algorithms, or principles (efficient reconstructive processes).

Views on Mastering the Number Combinations

One view of arithmetic learning that arose early in this century, was the "drill theory," a product of associative theories of learning. This theory assumed that (1) children must learn to imitate the skills and knowledge of adults; (2) what is learned are associations or bonds between otherwise unrelated stimuli; (3) understanding is not necessary for the formation of such bonds; and (4) the most efficient way to accomplish bond formation is drill (Brownell, 1935). Because drill theory viewed adult production of number facts as a reproductive process (e.g., the automatic association of two digits and their sum), the goal and method of instruction were clear. Children must form and strengthen bonds between two digits and, in addition, their sum

(Thorndike, 1922). Such links or associations are strengthened largely by means of repetition.[2]

Drill theory proponents did not consider children's counting strategies, discovery of relationships, or invented algorithms (devices for reasoning out sums, differences, etc.) important vehicles for mastering the number facts. Indeed, counting and invented algorithms were viewed as hindrances—as attempts to evade the real work of "memorizing" the number facts. For example, Wheeler (1939, p. 311) explained the relative difficulty of large number facts this way:

> As the size of the addend seems to be the general factor in causing the differences in the difficulty ranking, we wonder if the children are not computing the sums by physical or mental <u>counting</u>, a <u>crutch</u> which is probably developed in the child while building the number concepts [author's emphasis]. Psychologically the child should be able to learn 5 + 4 = 9 as easily as 2 + 3 = 5....

Similarly, Smith (1921, pp. 764-5) considered invented algorithms an impediment to learning the facts:

> Another pupil required a long time for the sum of 6 and 9. He explained his process as follows, "6 and 10 are 16; 6 and 9 are 1 less than 6 and 10; then 6 and 9 are 15." He had to think through a similar form every time any number was added to 9 and of course gave much slower responses.... We should be careful about letting pupils acquire forms or roundabout schemes for securing a result in the lower grades which will prove a handicap to them in the upper grades.

In sum, drill theory proposed that adult number fact knowledge (i.e., a network of automatic associations) is best achieved directly—by drill.

Early in this century, another—very different—account of arithmetic learning was advanced (see, for example, Brownell & Chazal, 1935; Buswell & Judd, 1925; Wheeler, 1935). According to Brownell (1935, p. 19):

> The "meaning" theory conceives arithm[illegible] closely knit system of understandable ideas, principles, and process[illegible] [illegible]rding to this theory, the test of learning is not mere mechanical faci[illegible] [illegible] "figuring." The true test is an intelligent grasp upon number relations and the ability to deal with arithmetical situations with proper comprehension of their mathematical as well as their practical significan[illegible]

In this view, drill, counting, the discovery of relationships, and the use of invented algorithms each have their place in learning the number combinations. Brownell noted that, initially, counting may be a necessary arithmetic strategy for children because it may be their only means of relating to numbers. As soon as they are ready, however, children should be introduced to more mature methods of dealing with arithmetic. Brownell suggested, for example, the algorithm of transforming an unfamiliar problem into a familiar one (e.g., 7 + 5: [7 + 3] + [5 - 3] = 10 + 2 = 12). Eventually, the child "comes to a confident knowledge of [a number combination], a knowledge full of meaning because of its frequent verification. By this time, the difficult stages of learning will long since have been passed, and habituation occurs rapidly and easily" (p. 24). Drill may serve to increase the facility and permanence of recall.

Meaning theory differs from drill theory in its view of number combination learning, then, in several important respects. First, learning mathematics—including mastery of the number combinations—is viewed as a slow, a protracted process. Children are not expected by meaning theory to imitate immediately the skill or knowledge of adults. In other words, the child's psychological readiness for learning is considered. At first children are expected to engage in immature strategies (the use of counting and later invented algorithms). Mature knowledge (including an appreciation of mathematical principles) then evolves from this experience. In sum, "children attain 'mastery' only after a period during which they deal with combinations by procedures less advanced (but to them more meaningful) than automatic responses"

(Brownell, 1941, p. 96). Second, because it arises from meaningful experience, habitual production of the number combinations has underlying meaning.

It appears that meaning theory is essentially correct about how the number combinations are learned. Counting experience is now viewed as an important basis for understanding arithmetic and performing it mentally (e.g., Gelman & Gallistel, 1978; Ginsburg, 1982, Resnick, 1983; Steffe, von Glasersfeld, Richards, & Cobb, 1983). The use of invented algorithms and other rules or principles are frequently advocated as aids in mastering the basic combinations (e.g., Cobb, 1983; Folsom, 1975; Rathmell, 1978; Trivett, 1980). Indeed, research (e.g., Brownell & Chazal, 1935; Thiele, 1938; Swenson, 1949) consistently demonstrates that teaching children "thinking strategies" is more effective than drill in facilitating learning, retention, and transfer of basic combinations (Suydam & Weaver, 1972).

Yet many educators still believe that learning the basic number combinations is essentially a straightforward, rote memory task that should be accomplished *quickly*. Indeed curriculum guides frequently overestimate how quickly children should master the combinations. For example, "Mathematics K-6: A Recommended Program for Elementary Schools" (The University of the State of New York/The State Education Department, Bureau of General Education Curriculum Development, Albany, NY 12234, 1980) includes mastery of the addition and subtraction facts (sums/minuends to 18) as an objective for the second level. (The third grade objectives is mastery of the addition and subtractions facts to 25). Such guidelines overlook the psychological evidence that mastery of basic addition and subtraction is often *not* achieved until third-grade or even later (e.g., Ashcraft, 1982; Woods, Resnick, & Groen, 1975) and that different groups (families) of combinations are *not* mastered in even a year's time (e.g., Baroody, 1983; Ginsburg & Baroody, 1983). Eleanor Duckworth (1982) wisely points out that most things worth knowing take a long time to learn and that teacher training needs to reinforce this point. Such advice is appropriate even when we consider the teaching and learning of basic skills such efficient production of number combinations.

Views on Mental Representation

It appears that advocates of meaning theory and the teaching of thinking strategies view the learning of rules, algorithms and principles as a means to an end: as vehicles for establishing specific numerical associations ("facts") in long-term memory. That is, over the course of development, children replace slow counting procedures and thinking strategies (inefficient reconstructive processes) with rapid fact retrieval (an efficient reproductive process) in order to do simple mental arithmetic (e.g., Ashcraft, 1982; Ilg & Ames, 1951; Resnick & Ford, 1981). In Brownell's terms, with experience, the production of the number combinations becomes habitual—a mechanical process that, is separate from but that can be related to a person's underlying semantic (meaning) system.

While a number of models (e.g., Ashcraft, 1982; Siegler & Shrager, 1983) have been proposed for how the number combinations are organized in an adult's memory, all share the assumption that some kind of reproductive process underlies production of the basic facts. Winkelman and Schmidt (1974) propose that addition and multiplication facts share a parallel organization. They argue that there are associations between two digits (e.g., 3 and 3) and both their sum (6) and product (9). As a result of associative interference, there is a greater tendency to associate 9 with 3 and 3 than, say, 7. Another model (Ashcraft & Battaglia, 1978; Ashcraft & Stazyk, 1981) proposes that the addition facts are mentally represented in memory as a printed table. The time needed to produce a particular fact is determined, in part, by the mental "distance" traversed during a memory search, i.e., the time needed to find the intersection of the two addends in the table. This helps to account for the slightly longer reaction times for problems with larger addends. Groen and Parkman (1972) suggest a direct access model in which all remembered facts are equally accessible.

Those facts that are not committed to memory are generated by the more immature counting-on (reconstructive) strategy. Resnick and Ford (1981, p. 74) give the following example to illustrate this model:

> If an adult were asked "How much is 3 + 4?" he or she would probably know immediately without really having to figure out the answer. Most adults have stored in long-term memory a response, 7, that is linked with the stimulus 3 + 4. It is as if there is a huge directory in their heads, and some of the entries were number facts that merely had to be "looked up." But think, now, what happens when a person has an occasional lapse of memory, when a number fact slips out of grasp. The answer is usually reconstructed in some way.

In brief, according to current theories, efficient production of number combinations is exclusively a reproductive process. Reconstructive processes are sometimes viewed as a less efficient back-up when a basic combination is not committed to long-term memory.

A number of chronometric studies have been undertaken, but no single model of mental representation has emerged as superior. Typically, differences among the models (structual variables such as min, sum, sum squared, etc.) are rarely, if ever, significant (Keven Miller, personnel communication, June 22, 1983). Moreover, though Ashcraft (1932) argues that his empirical evidence supports a table-like fact retrieval (sum-squared) model, the case for this (or any other association-based) model is not entirely convincing (Baroody, in press). Ashcraft uses a verification task (the subject is presented with an equation such as 5 + 3 = 9 and is required to respond true or false) to generate his data base. However, the verification task produces different results with older subjects than does a more straightforward production task (the subject is presented with a problem such as 5 + 3 and required to produce the answer). More specifically, zero problems tend to be verified inefficiently but produced efficiently.

Indeed, the reaction time latency curves for verification and production data typically differ in shape. It appears then that, because of the extra decision stage(s) required by the verification process, verification data may not accurately reflect memory search times and may not be the most suitable basis for drawing conclusions about the mental representation of number combinations.

Ashcraft, Fierman, and Bartolotta (in press) attempt to address the later criticism by reporting data that apparently shows that the verification and production tasks yield compatible results (parallel functions). The primary analysis (an ANOVA) involves the factors of grade, task and problem size. Unfortunately, problem size is analyzed in terms of a relatively crude small versus large sum breakdown. No justification is given for collapsing the data in a manner that might mask differences in the tasks.

Ashcraft et al. (in press) also notes that, while the slopes for his verification and production data are not parallel for previously used predictors (models), they are parallel when a new predictor (a revised retrieval model) is used. The revised model includes accessibility as a basic factor in retrieval of number combinations. To gauge this factor, Ashcraft proposes the use of a difficulty index. One index measure was their subjects' subjective ratings of problem difficulty. A second difficulty index was the Wheeler (1939) difficulty ratings. With the Wheeler variable as the predictor variable, the slopes for the verification and production data are indeed parallel. However, it is not clear why use of the Wheeler index is justified. Apparently, the Wheeler difficulty norms were based on the number of elementary students who "mastered" each fact—that is, produced the fact correctly and quickly. Unfortunately, the criterion of success was not more precisely defined. Moreover, the subjects in the Wheeler study were given a particular type of training. While the Wheeler index correlated well with a number of other difficulty indices of its day, there is no reason to believe it was or is a definitive measure of difficulty. After all, such indices are affected by the nature of the task, age of the subjects, criterion of success, the previous training of the subjects, etc. Unfortunately, data of Ashcraft's subjects' subjective difficulty ratings, which would seem a more appropriate index, were not reported. In brief, it still does not appear safe to assume that the verification data accurately reflects differences in the search/retrieval time of number combinations.

Clearly, one barrier in finding a clearly superior model of number combination representation/retrieval is the limitations of current methodology. Both production and verification data are "noisy." Both tasks assume a more or less unidirectional sequence of stages or processes (e.g., see Ashcraft, 1982), yet solving even simple problems may involve recursive processes (e.g., various checks). This may be especially true in ambiguous situations such as on the verification task where a subject is given a problem like 7 + 3 = 21. Indeed, because it introduces ambiguity and because it involves an extra decision stage, the verification task may be especially noisy. Moreover, the aim of chronometric analyses has been to uncover the way in which adults generate number combinations (cf. Siegler & Robinson, 1982). Yet detailed observation of adult arithmetic performance suggests that adults (like children) use a variety of strategies (see, e.g., Browne, 1906). In effect, adult chronometric data may reflect an averaging over different strategies rather than use of a single process (cf. Siegler & Robinson, 1982), and this may help to account for inconsistencies within the data of a single subject or sample and the inconsistencies among chronometric studies (cf. Baroody, in press).

It may be, then, that a clearly superior model has not emerged because current models are somehow inaccurate or incomplete. That is, the mental representation and efficient recall of number combinations may be more elaborate than simple associative models allow (Baroody, 1983). Because adults are flexible information processors, they may use several means—including reconstructive processes—to efficiently generate number combinations (Baroody & Ginsburg, 1982). It may be that some combinations are generated quite quickly from stored informal algorithms. Many others may be produced rapidly and directly from rules or principles that form an adult's mathematical semantic (meaning) system. By exploiting already internalized regularities or relationships, the child eliminates the need to learn and store hundreds of individual numerical associations. In sum, utilizing stored algorithms, rules or principle to quickly construct a range of combinations is cognitively more economical than relying exclusively on a network of individually stored facts.

Take, for example, the N + 0/0 + N family of combinations: 1 + 0, 2 + 0...10 + 0/0 + 1, 0 + 2...0 + 10. I worked with one kindergarten girl who was puzzled by 6 + 0 and finally responded "60." After helping her see that 0 was another name for nothing, she answered 6 + 0 correctly. Later in the session, she immediately responded to 3 + 0 with "3." A week later—without further intervention—she correctly and automatically responded to 0 + 5, 3 + 0, 4 + 0, 7 + 0, 6 + 0 and 0 + 8 (non-zero problems were interspersed). It appears that the child initially had an informal identity rule: "Nothing added to a set does not change the set." When she was introduced to the term "zero" and the written symbol 0, she assimilated this formal mathematics in terms of informal identity rule. The result was a formal N + 0/0 + N = N rule: "When zero and a number are added, the sum is the number." By using this abstracted rule, the child then appeared able to answer any N + 0/0 + N problem quickly and accurately (cf. Miller & Wellman, 1984; Thiele, 1938). How else can the transfer to the 3 + 0, or 5, etc. trials be explained? In all probability, she had never been exposed to—let alone practiced the combination "3 + 0 is 3 or "0 + 5 is 5". Thus it is not clear how association/fact retrieval models can account for such behavior.

In contrast to current models that posit associations among specific numbers, then, the alternative model allows that efficient generation of number combinations is due, <u>in part</u>, to storing and using algabraic or verbal labels. Thus instead of forming and storing individual associations for 3 + 0 and 3, 0 + 5 and 5, 88 + 0 and 88, and 0 +

1,000,000,000 and 1,000,000,000 etc.; the child may abstract a relationship and summarize the relationship in algebraic terms (N + 0 = N/0 + N = N) or use a (verbal) rule ("When zero and a number are added, the sum is the number"). Then when new problems are introduced, the most relevant algebraic expression or label is sought and used to reconstruct an answer. Such an economical process would make sense with an infinitely large number system.

In the case of zero combinations (N + 0 = N/0 + N = N, N - 0 = N, and N x 0 = 0/0 x N = 0), the algebraic or verbal) rules are relatively easy to learn. This would help to account for the observations that zero combinations are mastered relatively early (e.g., Groen & Parkman, 1972; Woods, Resnick, & Groen, 1975). Yet because of the form in which they are stored, the zero combinations may be particularly susceptible to performance failures (Type II errors). That is, because the algebraic or verbal rules are so similiar, they are relatively easy to confuse. While he referred to "designs" rather than rules, Thyne (1954) observed 20 years ago that "these designs would seem to be very 'easy' to learn—a possibility which might be expressed paradoxically by saying that is the very 'easiness' of the zero facts which makes them so 'difficult'. In other words, even young pupils can soon acquire a knowledge of how to answer zero facts in a way which is at once most consistent and most unreliable" (p. 205). Indeed, confusion in selecting among the rules is especially likely to occur in verification situations, where the stimulus (e.g., 5 x 0 = 5) may trigger two conflicting rules (N x 0 = 0 and N $\pm$ 0 = N).

More recently, Ashcraft (1983) has allowed that reconstructive processes may play some role in the efficient production of number combinations but that this is limited to the special cases involving 0 and 1. Specifically, he is willing to grant that children may learn an N - 0 = N rule and use well-learned "just before" count sequence relationships to generate solutions to N - 1 problems. But there are other possibilities for rule-governed production of basic subtraction combination. Children may quickly learn a N - N = 0 rule or identity principle to efficiently deal with problems such as 2 - 2, 9 - 9, 86 - 86. For problems with terms that differed by one (e.g., 6 - 5, 7 - 6, 8 - 7 or even 106 - 105), the child might realize that the answer is always one ("The subtraction of 'number neighbors' produces a difference of one"). Some adults may continue to use a nine rule: "A teen N minus 9 is N + 1" (e.g., 16 - 9 = 7, 17 - 9 = 8, 18 - 9 = 9). Finally, it is not implausible that some subtraction combinations may be efficiently reconstructed from addition counterparts (e.g., 10 - 7 is 3 because 7 + 3 is 10). In brief, efficient reconstructive processes may play a role in more than the efficient production of 0 and 1 combinations.

The teaching or encouragement of thinking strategies may, then, have a more direct bearing on mastering the basic combinations than is currently allowed. Such as instructional approach may not only help children to form specific numerical associations, it may help children to make rules algorithms and principles more explicit. As a result, algebraic or verbal labels may be internalized more readily and hence whole families of combinations may be learned more quickly.

Summary

In summary, in contrast to current models that view representation of the basic number combinations as a network of hundreds of specific numerical associations, it would seem cognitively more economical to mentally represent many groups or families of combinations in algebraic or verbal terms: as rules, algorithms or principles from which a whole range of combinations could be reconstructed. According to this alternative model, "mastery of the facts" would include discovering, labeling, and internalizing relationships. Meaningful instruction (the teaching of thinking strategies) contributes directly to this process and clearly would seem more efficient for this purpose than a drill approach. Thus, in contrast to current models that hypothesize reproductive processes replacing (slow) reconstructive processes, the alternative model suggests that some of the reconstructive processes involved in learning the combinations originally may continue to operate in adults, though more automatically.

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References

Ashcraft, M. H. (1982). The development of mental arithmetic: A chronometric approach. Developmental Review, 2, 213-236.

Ashcraft, M. H. (1983). Procedural knowledge versus fact retrieval in mental arithmetic: A reply to Baroody. Developmental Review, 3, 231-235.

Ashcraft, M. H., & Battaglia, J. (1978). Cognitive arithmetic: Evidence for retrieval and decision processes in mental addition. Journal of Experimental Psychology: Human Learning and Memory, 4, 527-538.

Ashcraft, M. H., Fierman, B. A., & Bartolotta, R. (in press). The production and verification tasks in mental addition: An empirical comparison. Developmental Review.

Ashcraft, M. H. & Stazyk, E. H. (1981). Mental addition: A test of three verification models. Memory and Cognition, 9, 185-196.

Baroody, A. J. (1983). The development of procedural knowledge: An alternative explanation for chronometric trends of mental arithmetic. Developmental Review, 3, 225-230.

Baroody, A. J. (in press). A re-examination of mental arithmetic models and data: A reply to Ashcraft. Developmental Review.

Baroody, A. J., & Ginsburg, H. P. (1982). Generating number combinations: Rote process or problem solving? Problem Solving, 4(12), 3-4.

Browne, C. E. (1906). The psychology of the simple arithmetical processes: A study of certain habits of attention and association. The American Journal of Psychology, XVII(1), 2-37.

Brownell, W. A. (1935). Psychological considerations in the learning and the teaching of arithmetic. The teaching of arithmetic (Tenth Yearbook, National Council of Teachers of Mathematics). New York: Bureau of Publications, Teachers College, Columbia University.

Brownell, W. A., & Chazal, C. (1935). The effects of premature drill in third-grade arithmetic. Journal of Educational Research, 29, 17-28.

Buswell, G. T., & Judd, C. H. (1925). Summary of educational investigations relating to arithmetic. Supplementary Educational Monographs, (No. 27).

Cobb, P. (1983, April). Children's construction of thinking strategies to find sums and differences. Paper presented at the annual meeting of the American Educational Research Association, Montreal.

Duckworth, E. (1982). A case study about some depths and perplexities of elementary arithmetic. In J. Bamberger & E. Duckworth (Eds.), An analysis of data from an experiment in teacher development (Report to NIE on grant number G81-0042).

Folsom, M. (1975). Operations on whole numbers. In J. N. Payne (Ed.), Mathematics learning in early childhood (37th Yearbook of the National Council of Teachers of Mathematics) (pp. 162-190). Reston, VA: NCTM.

Gelman, R., & Gallistel, C. R. (1978). The child's understanding of number. Cambridge, MA: Harvard University Press.

Ginsburg, H. P. (1982). Children's arithmetic. Austin, TX: Pro-Ed.

Greeno, J. G. (1978). Understanding and procedural knowledge in mathematics education. Educational Psychologist, 12 (3), 262-283.

Groen, G. J., & Parkman, J. M. (1972). A chronometric analysis of simple addition. Psychological Review, 79, 329-343.

Jerman, M. (1970). Some strategies for solving simple multiplication combinations. Journal for Research in Mathematics Education, 1, 95-128.

Kraner, R. E. (1980). Math deficits of learning disabled first graders with mathematics as a primary and secondary disorder. Focus on Learning Problems in Mathematics, 2 (3), 7-27.

Miller, K. F., & Wellman, H. (1984, April). Learning the rules of arithmetic: Children's understanding of zero. Paper presented at the annual meeting of the American Educational Research Association, Montreal.

Rathmell, E. C. (1978). Using thinking strategies to teach basic facts. In M. Suydam & R. Reys (Eds.), Developing computational skills, 1978 NCTM Yearbook. Reston, VA: NCTM.

Resnick, L. B. (1983). A developmental theory of number understanding. In H. P. Ginsburg (Ed.), The development of mathematical thinking. New York: Academic Press.

Resnick, L. B., & Ford, W. W. (1981). The psychology of mathematics for instruction. Hillsdale, NJ: Erlbaum.

Siegler, R. S., & Robinson, M. (1982). The development of numerical understandings. In H. W. Reese & Lewis P. Lipsitt (Eds.), Advances in child development and behavior: Vol. I. (pp. 241-312). New York: Academic Press.

Siegler, R. S., & Shrager, J. (in press). Strategy choices in addition: How do children know what to do? In C. Sophian (Ed.), Origins of cognitive skills (From the 18th Annual Carnegie Symposium on Cognition, Pittsburgh).

Smith, J. H. (1921). Arithmetic combinations. The Elementary School Journal, June, 762-770.

Steffe, L. P., von Glasersfeld, E., Richards, J. & Cobb, P. (1983). Children's counting types. New York: Praeger.

Svenson, O. (1975). Analyses of time required by children for simple additions. Acta Psychologica, 35, 289-302.

Thiele, C. (1938). The contribution of generalization to the learning of the addition facts. New York: Bureau of Publications, Teachers College, Columbia University.

Thorndike, E. L. (1922). The psychology of arithmetic. New York: Macmillan.

Thyne, J. M. (1954). Patterns of errors in the addition facts. Edinburgh: Scottish Council for Research in Education.

Trivett, J. (1980). The multiplication table: To be memorized or mastered? For the Learning of Mathematics, 1, 21-25.

Wheeler, L. R. (1939). A comparative study of the difficulty of the 100 addition combinations. The Journal of Genetic Psychology, 54, 295-312.

Winkelman, J. H., & Schmidt, J. (1974). Associative confusion in mental arithmetic. Journal of Experimental Psychology, 102, 734-736.

Woods, S. S., Resnick, L. B., & Groen, G. J. (1975). An experimental test of five process models for subtraction. Journal of Educational Psychology, 67 (1), 17-21.

Footnotes

1. Basic number combinations will refer to the 121 most basic addition problems, including those with single digit addends (0 + 0 to 9 + 9) and those in the series, 10 + 0 to 10 + 10 (and their commuted pairs). It will also refer to the corresponding subtraction, multiplication and division problems. In general, the term number combinations rather than number facts will be used. Number facts connotes a mechanical or rote associative process, and this term will be used to denote that meaning. Since number combinations may be learned in a meaningful manner, this less prejudicial term is preferred by the authors (cf. Brownell, 1935).

2. Thorndike's (1922) association theory of number fact learning was actually more sophisticated than this basic model. He argued that frequency of practice was not sufficient to account for number fact learning. He argued that bonds should not be formed independently—that instruction should be organized so as to build upon earlier, related learning. In addition to readiness, Thorndike argued that internal factors such as interest play a role in learning the number facts. Moreover, he even appeared to advocate the learning of rules—albeit in other terms. For example, he noted that "the facts are best learned once for all as the habits '1 times k is the same as k' and 'k times 1 is the same as k'" (pp. 144-145). It appears that Thorndike was advocating the teaching of a rule for this group or family of combinations, rather than formation of bonds for each of the individual facts. It is not clear, however, whether he was also advocating the use of the commutativity principle or the learning of two rules (one for N + 1 and another for 1 x N).